SUGGESTED SOLUTIONS TO HOMEWORK 8

Exercise 1 (8.1.6)**.** Show that $\lim(\arctan nx) = (\pi/2)\text{sgn}x$ for $x \in \mathbb{R}$.

Proof. We claim that

$$\lim(\arctan nx) = \begin{cases} -\pi/2, & x < 0, \\ 0, & x = 0, \\ \pi/2, & x > 0. \end{cases}$$

Indeed, since $\arctan nx$ is a odd function, it suffices to consider the case $x > 0$. Let $x > 0$ and $\varepsilon > 0$, then for $n > [x^{-1}\tan(\pi/2 - \varepsilon)] + 1$, we have

$$nx > \tan(\frac{\pi}{2} - \varepsilon),$$

therefore

$$\frac{\pi}{2} - \varepsilon < \arctan nx < \frac{\pi}{2},$$

which implies that

$$\lim_{n\to\infty} \arctan nx = \frac{\pi}{2}.$$

□

Exercise 2 (8.1.16)**.** Show that if $a > 0$, then the convergence of the sequence in Exercise 6 is uniform on the interval $[a, \infty)$, but is not uniform on the interval $(0, \infty)$.

Proof. Let $\varepsilon > 0$, then for $n > [a^{-1}\tan(\pi/2 - \varepsilon)] + 1$, we have

$$nx > \tan(\frac{\pi}{2} - \varepsilon),$$

for $x \geq a$, therefore

$$\frac{\pi}{2} - \varepsilon < \arctan nx < \frac{\pi}{2},$$

which implies that

$$\arctan nx \rightrightarrows \frac{\pi}{2} \text{ on } [a, \infty).$$

Consider $x_n = 1/n$, then

$$|\arctan nx_n - \frac{\pi}{2}\text{sgn}\frac{1}{n}| = |\frac{\pi}{4} - \frac{\pi}{2}\text{sgn}\frac{1}{n}| \geq \frac{\pi}{4},$$

which implies that the convergence is not uniform on $(0, \infty)$. □

Exercise 3 (8.1.19)**.** Show that the sequence $(x^2 e^{-nx})$ converges uniformly on $[0, \infty)$.

Proof. Let $\varepsilon > 0$, then for $n > [\sqrt{4e^{-2}\varepsilon^{-1}}] + 1$, we have

$$0 \leq x^2 e^{-nx} < \varepsilon,$$

for all $x \geq 0$, which implies that

$$x^2 e^{-nx} \rightrightarrows 0 \text{ on } [0, \infty].$$

□

Exercise 4 (8.1.23)**.** Let (f_n), (g_n) be sequences of bounded functions on A that converge uniformly on A to f, g, respectively. Show that $(f_n g_n)$ converges uniformly on A to fg.

Proof. Let $\varepsilon > 0$, since (f_n) and (g_n) uniformly converge to f and g respectively, there exists a $N \in \mathbb{N}$ such that for $n > N$, we have

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}, \quad |g_n(x) - g(x)| < \frac{\varepsilon}{2},$$

for $x \in A$. Moreover, since f_{N+1} and g_{N+1} are bounded on A, there exists a constant $C > 0$ such that

$$|f(x)| \leq |f_{N+1}(x)| + \frac{\varepsilon}{2} < C, \quad |g(x)| \leq |g_{N+1}(x)| + \frac{\varepsilon}{2} < C, \tag{1}$$

which implies that f and g are bounded. Then

$$|f_n(x)g_n(x) - f(x)g(x)| \leq |f_n(x) - f(x)| \cdot |g(x)| + |f(x)| \cdot |g_n(x) - g(x)| < C\varepsilon,$$

which implies that $(f_n g_n)$ converges uniformly on A to fg. □

Exercise 5 (8.2.9)**.** Let $f_n(x) := x^n/n$ for $x \in [0,1]$. Show that the sequence (f_n) of differentiable functions converges uniformly to a differentiable function f on $[0,1]$, and that the sequence (f'_n) converges on $[0,1]$ to a function g, but that $g(1) \neq f'(1)$.

Proof. We claim that (f_n) uniformly converges to 0 on $[0,1]$. Indeed, let $\varepsilon > 0$, then for $n > \varepsilon$, we have

$$0 \leq f_n(x) < \varepsilon,$$

for $x \in [0,1]$, which implies that

$$f_n \rightrightarrows 0 \text{ on } [0,1].$$

In addition, we claim that (f'_n) converges to g on $[0,1]$, where g is defined as

$$g(x) := \begin{cases} 0, & 0 \leq x < 1, \\ 1, & x = 1. \end{cases}$$

Indeed, for $x \in [0,1)$, let $\varepsilon > 0$, for $n > x^{-n+1}$, we have

$$0 \leq f'_n(x) < \varepsilon,$$

which implies that f'_n converges to 0 on $[0,1)$. Moreover, Since $f'_n(x) = x^{n-1}$, we have (f'_n) converges to 1 at $x = 1$.

Therefore it is clear that $g(1) \neq f'(1)$. □

Exercise 6 (8.2.12)**.** Show that $\lim \int_1^2 e^{-nx^2} dx = 0$.

Proof. We claim that (e^{-nx^2}) uniformly converges to 0 on $[1,2]$. Therefore

$$\lim_{n\to\infty} \int_1^2 e^{-nx^2} dx = 0.$$

Indeed, let $\varepsilon > 0$, then for $n > \ln \varepsilon$, we have

$$0 < e^{-nx^2} < \varepsilon,$$

which implies that (e^{-nx^2}) uniformly converges to 0 on $[1,2]$. □